Lesson 9.4 Exercises, pages 665–671

Exercises

A 3. For each expression below:
   i) Determine any non-permissible values of $\theta$.
   ii) Write the expression as a single term.

   a) $1 - \cos^2 \theta$
      i) All real values are permissible.
      ii) $1 - \cos^2 \theta = \sin^2 \theta$

   b) $\cos^2 \theta - 1$
      i) All real values are permissible.
      ii) $\cos^2 \theta - 1 = -\sin^2 \theta$

   c) $\sec^2 \theta - 1$
      i) $\sec^2 \theta = \frac{1}{\cos^2 \theta}$, so $\cos \theta \neq 0$,
         $\theta \neq \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$
      ii) $\sec^2 \theta - 1 = \tan^2 \theta$

   d) $1 - \sec^2 \theta$
      i) $\sec^2 \theta = \frac{1}{\cos^2 \theta}$
         so $\cos \theta \neq 0$,
         $\theta \neq \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$
      ii) $1 - \sec^2 \theta = -\tan^2 \theta$

   e) $\frac{\cot^2 \theta + 1}{\cot^2 \theta}$
      i) $\cot^2 \theta \neq 0$
         $\theta \neq \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$
      ii) $\frac{\cot^2 \theta + 1}{\cot^2 \theta} = \frac{\csc^2 \theta}{\cot^2 \theta}$
         $\frac{1}{\sin^2 \theta}$
         $= \frac{\cot^2 \theta}{\cos^2 \theta}$
         $= \frac{1}{\cos^2 \theta}$
         $= \sec^2 \theta$

   f) $\csc^2 \theta - \cot^2 \theta$
      i) $\csc^2 \theta = \frac{1}{\sin^2 \theta}$ and
         $\cot^2 \theta = \frac{\cos^2 \theta}{\sin^2 \theta}$, so
         $\sin \theta \neq 0, \theta \neq \pi k, k \in \mathbb{Z}$
      ii) $\csc^2 \theta - \cot^2 \theta = 1$
4. a) Verify the identity \( \tan^2\theta + 1 = \sec^2\theta \) for \( \theta = \frac{2\pi}{3} \).

\[
\begin{align*}
\text{L.S.} &= \tan^2\theta + 1 \\
&= \tan^2\left(\frac{2\pi}{3}\right) + 1 \\
&= (-\sqrt{3})^2 + 1 \\
&= 4 + 1 \\
&= 5
\end{align*}
\]

\[
\begin{align*}
\text{R.S.} &= \sec^2\theta \\
&= \sec^2\left(\frac{2\pi}{3}\right) \\
&= \left(\frac{1}{\cos\theta}\right)^2 \\
&= \frac{1}{\cos^2\left(\frac{2\pi}{3}\right)} \\
&= \frac{1}{\left(-\frac{1}{2}\right)^2} \\
&= 4
\end{align*}
\]

The left side is equal to the right side, so the identity is verified.

b) Verify the identity \( 1 + \cot^2\theta = \csc^2\theta \) using graphing technology.

The graphs of \( y = 1 + \frac{1}{\tan^2\theta} \) and \( y = \frac{1}{\sin^2\theta} \) coincide, so the identity is verified.

c) Verify the identity \( \sin^2\theta + \cos^2\theta = 1 \) for \( \theta = 300^\circ \).

\[
\begin{align*}
\text{L.S.} &= \sin^2\theta + \cos^2\theta \\
&= \sin^2(300^\circ) + \cos^2(300^\circ) \\
&= \left(-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \\
&= 1 + \frac{1}{4} \\
&= \frac{5}{4} \\
&= \text{R.S.}
\end{align*}
\]

The left side is equal to the right side, so the identity is verified.
5. For each expression below:

i) Determine any non-permissible values of \( \theta \).

ii) Write the expression as a single term.

\[ \sqrt{\frac{1 - \sin^2 \theta}{1 + \tan^2 \theta}} \]

i) \( \tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta} \) so 
\[ \cos \theta \neq 0, \theta \neq \frac{\pi}{2} + \pi k, k \in \mathbb{Z} \]

ii) 
\[ \frac{\sqrt{\cos^2 \theta}}{\sqrt{\sec^2 \theta}} \]
\[ = \frac{|\cos \theta|}{1} \]
\[ = |\cos \theta|^2 \]
\[ = \cos^2 \theta \]

\[ \frac{\cos \theta}{1 + \sin \theta} + \frac{\cos \theta}{1 - \sin \theta} \]

i) \( \sin \theta \neq \pm 1, \)
so \( \theta \neq \frac{\pi}{2} + \pi k, k \in \mathbb{Z} \)

ii) 
\[ \frac{\cos \theta(1 - \sin \theta) + \cos \theta(1 + \sin \theta)}{(1 + \sin \theta)(1 - \sin \theta)} \]
\[ = \frac{\cos \theta - \cos \theta \sin \theta + \cos \theta + \cos \theta \sin \theta}{1 - \sin^2 \theta} \]
\[ = \frac{2 \cos \theta}{\cos^2 \theta} \]
\[ = 2 \sec \theta \]

\[ \frac{\csc \theta}{\cot \theta + \tan \theta} \]

i) \( \tan \theta = \frac{\sin \theta}{\cos \theta} \) so \( \cos \theta \neq 0, \)
\( \theta \neq \frac{\pi}{2} + \pi k, k \in \mathbb{Z} \); \( \csc \theta = \frac{1}{\sin \theta} \) and
\( \cot \theta = \frac{\cos \theta}{\sin \theta} \), so \( \sin \theta \neq 0, \theta \neq \pi k, k \in \mathbb{Z} \)

ii) 
\[ \frac{\csc \theta}{\cot \theta + \tan \theta} \]
\[ = \frac{1}{\sin \theta} \]
\[ = \frac{\cos \theta}{\sin \theta + \cos \theta} \]
Multiply numerator and denominator by \( \cos \theta \sin \theta \).
\[ = \frac{\cos \theta}{\cos^2 \theta + \sin^2 \theta} \]
\[ = \frac{\cos \theta}{1} \]
\[ = \cos \theta \]
6. For each identity:
   i) Verify the identity using graphing technology.
   ii) Show that:
      a) \(1 - \cos^2 \theta = \cos^2 \theta \tan^2 \theta\)
         i) The graphs of \(y = 1 - \cos^2 \theta\)
         and \(y = \cos^2 \theta \tan^2 \theta\) coincide,
         so the identity is verified.
         ii) R.S. = \(\cos^2 \theta \tan^2 \theta\)
              = \(\cos^2 \theta \left(\frac{\sin^2 \theta}{\cos^2 \theta}\right)\)
              = \(\sin^2 \theta\)
              = \(1 - \cos^2 \theta\)
              = L.S.
      b) \(\sin^2 \theta + \cos^2 \theta + \tan^2 \theta = \sec^2 \theta\)
         i) The graphs of \(y = \frac{1}{\cos^2 \theta}\)
         and \(y = \sin^2 \theta + \cos^2 \theta + \tan^2 \theta\)
         coincide, so the identity is verified.
         ii) L.S. = \(\sin^2 \theta + \cos^2 \theta + \tan^2 \theta\)
              = \(1 + \tan^2 \theta\)
              = \(\sec^2 \theta\)
              = R.S.

7. Use algebra to solve each equation over the domain \(-\pi \leq x \leq \pi\).
   a) \(\sin^2 x + 2 \cos x = 2\)
      \((1 - \cos^2 x) + 2 \cos x = 2\)
      \(-\cos^2 x + 2 \cos x - 1 = 0\)
      \(\cos^2 x - 2 \cos x + 1 = 0\)
      \((\cos x - 1)^2 = 0\)
      \(\cos x = 1\)
      \(x = 0\)
      The root is: \(x = 0\)
      Verify by substitution.
   b) \(2 \cos^2 x - \sin x - 1 = 0\)
      \(2(1 - \sin^2 x) - \sin x - 1 = 0\)
      \(2 - 2 \sin^2 x - \sin x - 1 = 0\)
      \(2 \sin^2 x + \sin x - 1 = 0\)
      \((2 \sin x - 1)(\sin x + 1) = 0\)
      Either \(2 \sin x - 1 = 0\) or \(\sin x + 1 = 0\)
      \(2 \sin x = 1\)
      \(\sin x = \frac{1}{2}\)
      \(x = \frac{\pi}{2}, x = \frac{5\pi}{6}\)
      The roots are: \(x = \frac{\pi}{2}, x = \frac{5\pi}{6}\)
      Verify by substitution.

8. Is either of these statements true? Justify your answer.
   a) Since \(\tan \theta = \frac{\sin \theta}{\cos \theta}\), then \(\tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta}\)
      This statement is true because if \(c = \frac{a}{b}\), then I can square both sides to obtain \(c^2 = \frac{a^2}{b^2}\).
   b) Since \(\sin^2 \theta + \cos^2 \theta = 1\), then \(\sin \theta + \cos \theta = 1\)
      This statement is false. For example, for \(\theta = \frac{\pi}{4}\), \(\sin^2 \left(\frac{\pi}{4}\right) + \cos^2 \left(\frac{\pi}{4}\right) = 1,\)
      but \(\sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\) or \(\frac{2}{\sqrt{2}}\), which is not 1.
9. Solve each equation, then write the general solution.

a) \(4 \sin^2 x + 2 \tan^2 x = 2 \sec^2 x\)

\[
4 \sin^2 x + 2 \tan^2 x - 2 \sec^2 x = 0
\]

\[
4 \sin^2 x - 2(\sec^2 x - \tan^2 x) = 0
\]

\[
4 \sin^2 x - 2 = 0
\]

\[
4 \sin^2 x = 2
\]

\[
\sin^2 x = \frac{1}{2}
\]

\[
\sin x = \pm \sqrt{\frac{1}{2}}
\]

\[
x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}
\]

Verify by substitution.
The general solution is: \(x = \frac{\pi}{4} + \frac{\pi}{2} k, k \in \mathbb{Z}\)

b) \(\csc^2 x + \tan^2 x - \cot^2 x = 4\)

\[
\csc^2 x - \cot^2 x + \tan^2 x = 4
\]

\[
1 + \tan^2 x = 4
\]

\[
\tan^2 x = 3
\]

\[
\tan x = \pm \sqrt{3}
\]

\[
x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}
\]

Verify by substitution.
The general solution is: \(x = \frac{\pi}{3} + \pi k, k \in \mathbb{Z}\)

10. Use algebra to solve each equation over the domain 
\(-90^\circ \leq x \leq 270^\circ\). Give the roots to the nearest degree.

a) \(4 - 4 \cos^2 x = \sin x\)

\[
4 - 4(1 - \sin^2 x) - \sin x = 0
\]

\[
4 - 4 + 4 \sin^2 x - \sin x = 0
\]

\[
4 \sin^2 x - \sin x = 0
\]

\[
(\sin x)(4 \sin x - 1) = 0
\]

Either \(\sin x = 0\)

\[
x = 0^\circ, x = 180^\circ
\]

Or \(4 \sin x - 1 = 0\)

\[
4 \sin x = 1
\]

\[
\sin x = \frac{1}{4}
\]

\[
x \approx 14^\circ, x \approx 166^\circ
\]

The roots are: \(x = 0^\circ, x = 180^\circ, x \approx 14^\circ, x \approx 166^\circ\)

Verify by substitution.

b) \(\cos x + 1 = 2 \sin^2 x\)

\[
\cos x + 1 - 2(1 - \cos^2 x) = 0
\]

\[
\cos x + 1 - 2 + 2 \cos^2 x = 0
\]

\[
2 \cos^2 x + \cos x - 1 = 0
\]

\[
(2 \cos x - 1)(\cos x + 1) = 0
\]

Either \(2 \cos x - 1 = 0\)

\[
2 \cos x = 1
\]

\[
\cos x = \frac{1}{2}
\]

\[
x = 60^\circ, x = -60^\circ
\]

Or \(\cos x + 1 = 0\)

\[
\cos x = -1
\]

\[
x = 180^\circ
\]

The roots are: \(x = 60^\circ, x = -60^\circ, x = 180^\circ\)

Verify by substitution.
11. Identify whether each equation is an identity. Justify your answer.

Prove each identity. Use algebra to solve each equation that is not an identity over the domain \(-\pi \leq x \leq \pi\). Give the roots to the nearest hundredth.

\(a\) \(\cos^2 x = (\sin x)(\csc x + \sin x)\)

The graphs of \(y = \cos^2 x\) and \(y = (\sin x)\left(\frac{1}{\sin x} + \sin x\right)\) do not coincide, so the equation is not an identity.

\[\cos^2 x = (\sin x)\left(\frac{1}{\sin x} + \sin x\right), \quad \sin x \neq 0\]

\[\cos^2 x = 1 + \sin^2 x\] Substitute: \(\cos^2 x = 1 - \sin^2 x\)

\[1 - \sin^2 x = 1 + \sin^2 x\]

\[2 \sin^2 x = 0\]

\[\sin^2 x = 0\]

\[\sin x = 0\]

Since this is a non-permissible value, the equation has no real solution.

\(b\) \((\cos x)(\sec x - \cos x) = \cos^2 x\)

The graphs of \(y = (\cos x)\left(\frac{1}{\cos x} - \cos x\right)\) and \(y = \cos^2 x\) do not coincide, so the equation is not an identity.

\((\cos x)(\sec x) - \cos^2 x = \cos^2 x, \quad \cos x \neq 0\)

\[1 - 2 \cos x = 0\]

\[\cos^2 x = \frac{1}{2}\]

\[\cos x = \pm \sqrt{\frac{1}{2}}\]

The roots are: \(x \approx 0.79, x \approx 2.36, x \approx -2.36, x \approx -0.79\)

Verify by substitution.

12. Solve each equation, then write the general solution.

\(a\) \(2 \cos^2 x + \tan^2 x = 2\)

\[2 \cos^2 x + \sec^2 x - 1 = 2\]

\[2 \cos^2 x + \frac{1}{\cos^2 x} - 1 = 2, \quad \text{assume } \cos^2 x \neq 0\]

\[2 \cos^2 x + \frac{1}{\cos^2 x} - 3 = 0\]

\[2 \cos^2 x + 1 - 3 \cos^2 x = 0\]

\[2 \cos^2 x - 3 \cos^2 x + 1 = 0\]

\[(2 \cos^2 x - 1)(\cos^2 x - 1) = 0\]

Either \(2 \cos^2 x - 1 = 0\) or \(\cos^2 x - 1 = 0\)

\[2 \cos^2 x = 1\]

\[\cos^2 x = \frac{1}{2}\]

\[\cos x = \pm \sqrt{\frac{1}{2}}\]

\[x = \frac{\pi}{4}, \quad \frac{3\pi}{4}, \quad \frac{5\pi}{4}, \quad \frac{7\pi}{4}\]

Verify by substitution.

The general solution is: \(x = \frac{\pi}{2} + \pi k, \quad k \in \mathbb{Z}\)

\(b\) \(\sin^2 x + \cot^2 x - 1 = 0\)

\[\sin^2 x + \csc^2 x - 1 - 1 = 0\]

\[\sin^2 x + \csc^2 x - 2 = 0\]

\[\sin^2 x + \frac{1}{\sin^2 x} - 2 = 0\]

\[
\sin^2 x + 1 - 2 \sin^2 x = 0, \text{ assume } \sin^2 x \neq 0
\]

\[\sin^2 x - 2 \sin^2 x + 1 = 0\]

\[(\sin^2 x - 1)^2 = 0\]

\[\sin^2 x = 1\]

\[\sin x = \pm 1\]

\[x = \frac{\pi}{2}, \quad \frac{3\pi}{2}\]

Verify by substitution.

The general solution is: \(x = \frac{\pi}{2} + \pi k, \quad k \in \mathbb{Z}\) or \(x = \pi k, \quad k \in \mathbb{Z}\)
13. Determine a single trigonometric function for $m$ such that the equation $\frac{2 - \sin^2 \theta}{\cos \theta} = m + \cos \theta$ is an identity. Verify your answer by proving the identity.

\[
\frac{2 - \sin^2 \theta}{\cos \theta} = m + \cos \theta \quad \text{Solve for } m.
\]

\[
m = \frac{2 - \sin^2 \theta}{\cos \theta} - \cos \theta \quad \text{Use a common denominator.}
\]

\[
= \frac{2 - \sin^2 \theta - \cos^2 \theta}{\cos \theta}
\]

\[
= \frac{2 - (\sin^2 \theta + \cos^2 \theta)}{\cos \theta}
\]

\[
= \frac{2 - 1}{\cos \theta}
\]

\[
= \frac{1}{\cos \theta}
\]

\[
= \sec \theta
\]

The identity is: $\frac{2 - \sin^2 \theta}{\cos \theta} = \sec \theta + \cos \theta$

L.S. = $\frac{2 - \sin^2 \theta}{\cos \theta}$

\[
= \frac{2 - (1 - \cos^2 \theta)}{\cos \theta}
\]

\[
= \frac{1 + \cos^2 \theta}{\cos \theta}
\]

\[
= \frac{1}{\cos \theta} + \frac{\cos^2 \theta}{\cos \theta}
\]

\[
= \sec \theta + \cos \theta
\]

= R.S.

Since the left side is equal to the right side, the identity is proved.