Lesson 7.6 Exercises, pages 451–455

Exercises

3. Approximate the value of each logarithm, to the nearest thousandth.
   a) \( \log_2 9 \)  
   b) \( \log_2 100 \)  
   c) \( \log_2 1000 \)

   Use the change of base formula to change the base of the logarithms to base 10.

   \[
   \log_2 9 = \frac{\log 9}{\log 2} = 3.1699... \approx 3.170
   \]

   \[
   \log_2 100 = \frac{\log 100}{\log 2} = 6.6438... \approx 6.644
   \]

   \[
   \log_2 1000 = \frac{\log 1000}{\log 2} = 9.9657... \approx 9.966
   \]

4. Order these logarithms from greatest to least:
   \( \log_2 80, \log_3 900, \log_4 5000, \log_5 10000 \)

   Write each logarithm to base 10, then calculate its value.

   \[
   \log_2 80 = \frac{\log 80}{\log 2} = 6.3219... \approx 6.322
   \]

   \[
   \log_3 900 = \frac{\log 900}{\log 3} = 6.1918... \approx 6.192
   \]

   \[
   \log_4 5000 = \frac{\log 5000}{\log 4} = 6.1438... \approx 6.144
   \]

   \[
   \log_5 10000 = \frac{\log 10000}{\log 5} = 5.7227... \approx 5.723
   \]

   From greatest to least: \( \log_2 80, \log_3 900, \log_4 5000, \log_5 10000 \)
5. Approximate the value of each logarithm, to the nearest thousandth. Write the related exponential expression.

a) \( \log_7 400 \)

\[
\log_{7} 400 = \frac{\log 400}{\log 7} = 3.079 \ldots \\
\approx 3.079
\]

So, \( 400 \approx 7^{3.079} \)

b) \( \log_{\frac{1}{2}} \frac{1}{2} \)

\[
\log_{\frac{1}{2}} \frac{1}{2} = \frac{\log 0.5}{\log 3} = -0.6309 \ldots \\
\approx -0.631
\]

So, \( \frac{1}{2} \approx 3^{-0.631} \)

6. a) Use technology to graph \( y = \log_5 x \). Sketch the graph.

Graph: \( y = \frac{\log x}{\log 5} \)

b) Identify the intercepts and the equation of the asymptote of the graph, and the domain and range of the function.

From the graph, the \( x \)-intercept is 1. There is no \( y \)-intercept.
The equation of the asymptote is \( x = 0 \).
The domain of the function is \( x > 0 \).
The range of the function is \( y \in \mathbb{R} \).

c) Choose the coordinates of two points on the graph.

Multiply their \( x \)-coordinates and add their \( y \)-coordinates.
What do you notice about the new coordinates? Explain the result.

From the table of values, two points on the graph have coordinates: (5, 1) and (25, 2)
The product of the \( x \)-coordinates is 125. The sum of the \( y \)-coordinates is 3.
The new coordinates are (125, 3), which is also a point on the graph.
The logarithm of the product of two numbers is the sum of the logarithms of the numbers.

7. a) Use graphing technology to graph \( y = \log_2 x \), \( y = \log_4 x \), and \( y = \log_8 x \). Sketch the graphs.

Graph: \( y = \frac{\log x}{\log 2} \), \( y = \frac{\log x}{\log 4} \), and \( y = \frac{\log x}{\log 8} \)
b) In part a, what happened to the graph of \( y = \log_b x \), \( b > 0 \), \( b \neq 1 \), as the base changed?

As \( b \) increases, from \( b = 2 \), the graph of \( y = \log_2 x \) is compressed vertically by a factor of:

\[
\log x \quad \frac{\log b}{\log 2} = \frac{\log 2}{\log b}
\]

8. a) The graphs of a logarithmic function and its transformation image are shown. The functions are related by translations, and corresponding points are indicated. Identify the translations.

From A to A’, the translations are 3 units left and 1 unit up.
The same translations relate B and B’.

b) Given that \( f(x) = \log_b x \), what is \( g(x) \)? Justify your answer.

After translations, the image of the graph of \( y = \log_b x \) has equation:

\[ y - k = \log_b(x - h) \quad \text{Substitute: } k = 1 \text{ and } h = -3 \]

The image graph has equation \( y - 1 = \log_b(x + 3) \); or \( y = \log_b(x + 3) + 1 \)

So, \( g(x) = \log_b(x + 3) + 1 \)

9. a) How is the graph of \( y = 2 \log_2(2x - 8) \) related to the graph of \( y = \log_2 x \)? Sketch both graphs on the same grid.

Compare \( y = 2 \log_2(x - 4) \) with \( y - k = c \log_b(x - h) \):

\[ k = 0, \quad c = 2, \quad d = 2, \quad \text{and } h = 4 \]

Write \( y = 2 \log_2(2x - 8) \) as \( y = 2 \log_2(x - 4) \).

The graph of this function is the image of the graph of \( y = \log_2 x \) after a vertical stretch by a factor of 2, a horizontal compression by a factor of \( \frac{1}{2} \), then a translation of 4 units right.

Use the general transformation: \((x, y)\) corresponds to \((\frac{x}{a} + h, cy + k)\)

The point \((x, y)\) on \( y = \log_2 x \) corresponds to the point \( (\frac{x}{2} + 4, 2y) \) on \( y = 2 \log_2(2x - 8) \).

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>((\frac{x}{2} + 4, 2y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.5, -1)</td>
<td>(4.25, -2)</td>
</tr>
<tr>
<td>(1, 0)</td>
<td>(4.5, 0)</td>
</tr>
<tr>
<td>(2, 1)</td>
<td>(5, 2)</td>
</tr>
<tr>
<td>(4, 2)</td>
<td>(6, 4)</td>
</tr>
<tr>
<td>(8, 3)</td>
<td>(8, 6)</td>
</tr>
</tbody>
</table>
b) Identify the intercepts and the equation of the asymptote of the graph of \(y = 2 \log_2(2x - 8)\), and the domain and range of the function. Use graphing technology to verify.

From the graph, there is no \(y\)-intercept.
From the table, the \(x\)-intercept is 4.5.
The equation of the asymptote is \(x = 4\).
The domain of the function is \(x > 4\).
The range of the function is \(y \in \mathbb{R}\).

10. a) Graph \(y = -\frac{1}{4} \log_2 \left( \frac{1}{2} x \right) + 1\).

Compare \(y - 1 = -\frac{1}{4} \log_2 \left( \frac{1}{2} x \right)\) with \(y - k = c \log_d(x - h)\):
\(k = 1, c = -\frac{1}{4}, d = \frac{1}{2}, \text{ and } h = 0\)
Use the general transformation:
\((x, y)\) corresponds to \((\frac{x}{d} + h, cy + k)\)
The point \((x, y)\) on \(y = \log_2 x\) corresponds to the point \((2x, -\frac{1}{4}y + 1)\)
on \(y = -\frac{1}{4} \log_2 \left( \frac{1}{2} x \right) + 1\).

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>((2x, -\frac{1}{4}y + 1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0.25, -2))</td>
<td>((0.5, 1.5))</td>
</tr>
<tr>
<td>((0.5, -1))</td>
<td>((1, 1.25))</td>
</tr>
<tr>
<td>((1, 0))</td>
<td>((2, 1))</td>
</tr>
<tr>
<td>((2, 1))</td>
<td>((4, 0.75))</td>
</tr>
<tr>
<td>((4, 2))</td>
<td>((8, 0.5))</td>
</tr>
</tbody>
</table>

b) Identify the intercepts and the equation of the asymptote of the graph of \(y = -\frac{1}{4} \log_2 \left( \frac{1}{2} x \right) + 1\), and the domain and range of the function.

From the graph, there is no \(y\)-intercept.
Use graphing technology to generate a table of values, or to graph the function. The \(x\)-intercept is 32.
The equation of the asymptote is \(x = 0\).
The domain of the function is \(x > 0\).
The range of the function is \(y \in \mathbb{R}\).
11. Graph the function \( y = -\frac{1}{3} \log_3(-2x - 4) + 5 \), then identify the intercepts, the equation of the asymptote, and the domain and range of the function.

Write \( y = -\frac{1}{3} \log_3(-2x - 4) + 5 \) as:
\[
y - 5 = -\frac{1}{3} \log_3[-2(x + 2)]
\]

Compare \( y - 5 = -\frac{1}{3} \log_3[-2(x + 2)] \) with \( y - k = c \log_d(x - h) \):
\[
k = 5, \quad c = -\frac{1}{3}, \quad d = -2, \quad \text{and} \quad h = -2
\]

Use the general transformation:
\( (x, y) \) corresponds to \( \left( \frac{x}{d} + h, cy + k \right) \)

The point \( (x, y) \) on \( y = \log_3 x \) corresponds to the point
\[
\left( -\frac{1}{2}x - 2, -\frac{1}{3}y + 5 \right)
\]
on \( y = -\frac{1}{3} \log_3(-2x - 4) + 5 \).

<table>
<thead>
<tr>
<th>( (x, y) )</th>
<th>( \left( -\frac{1}{2}x - 2, -\frac{1}{3}y + 5 \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \left( \frac{1}{3}, -2 \right) )</td>
<td>( \left( \frac{37}{18}, \frac{17}{3} \right) )</td>
</tr>
<tr>
<td>( \left( \frac{1}{3}, -1 \right) )</td>
<td>( \left( \frac{13}{6}, \frac{16}{3} \right) )</td>
</tr>
<tr>
<td>(1, 0)</td>
<td>( \left( \frac{5}{2}, 5 \right) )</td>
</tr>
<tr>
<td>(3, 1)</td>
<td>( \left( \frac{7}{2}, \frac{14}{3} \right) )</td>
</tr>
<tr>
<td>(9, 2)</td>
<td>( \left( \frac{13}{2}, \frac{13}{3} \right) )</td>
</tr>
</tbody>
</table>

From the graph, there is no \( y \)-intercept.
The equation of the asymptote is \( x = -2 \).
The domain of the function is \( x < -2 \).
The range of the function is \( y \in \mathbb{R} \).

To determine the \( x \)-intercept, solve the equation:
\[
0 = -\frac{1}{3} \log_3(-2x - 4) + 5
\]
\[
-5 = -\frac{1}{3} \log_3(-2x - 4)
\]
\[
15 = \log_3(-2x - 4) \quad \text{Write in exponential form.}
\]
\[
-2x - 4 = 3^{15}
\]
\[
-2x = 14348911
\]
\[
x = -7174455.5
\]