3. Which sequences could be geometric? If a sequence is geometric, state its common ratio.

a) 1, 2, 4, 8, 16, . . .
   The sequence is geometric.
   \( r = \frac{2}{1} = 2 \)

b) 4, 9, 16, 25, 36, . . .
   The sequence is not geometric.

c) –3, 2, 7, 12, 17, . . .
   The sequence is not geometric.

d) 6, 0.6, 0.06, 0.006, . . .
   The sequence is geometric.
   \( r = \frac{0.6}{6} = 0.1 \)

e) 10, 100, 1000, 10 000
   The sequence is geometric.
   \( r = \frac{100}{10} = 10 \)

f) 2, 4, 6, 8, 10, . . .
   The sequence is not geometric.

g) 192, –96, 48, –24, 12, . . .
   The sequence is geometric.
   \( r = \frac{–96}{192} = –\frac{1}{2} \)

h) 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, . . .
   The sequence is not geometric.
4. State the common ratio, then write the next 3 terms of each geometric sequence.

a) \(-1, -3, -9, \ldots\)
   \(r = \frac{-3}{-1} = 3\)
   The next 3 terms are: \(-27, -81, -243\)

b) \(48, 24, 12, \ldots\)
   \(r = \frac{24}{48} = 0.5\)
   The next 3 terms are: \(6, 3, 1.5\)

c) \(25, -50, 100, \ldots\)
   \(r = \frac{-50}{25} = -2\)
   The next 3 terms are: \(-200, 400, -800\)

d) \(4, -2, 1, \ldots\)
   \(r = \frac{-2}{4} = -0.5\)
   The next 3 terms are: \(-0.5, 0.25, -0.125\)

e) \(\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \ldots\)
   \(r = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}\)
   The next 3 terms are: \(\frac{1}{54}, \frac{1}{162}, \frac{1}{486}\)

f) \(-24, 12, -6, \ldots\)
   \(r = \frac{-24}{-12} = -1\)
   The next 3 terms are: \(3, -\frac{3}{2}, \frac{3}{4}\)

5. For each geometric sequence, determine the indicated value.

a) \(3, 6, 12, \ldots; \) determine \(t_7\)
   \(r = \frac{6}{3} = 2\)
   Use: \(t_n = t_1 r^{n-1}\)
   Substitute: \(n = 7, t_1 = 3, r = 2\)
   \(t_7 = 3(2)^7 = 384\)

b) \(18, 9, 4.5, \ldots; \) determine \(t_6\)
   \(r = \frac{9}{18} = 0.5\)
   Use: \(t_n = t_1 r^{n-1}\)
   Substitute: \(n = 6, t_1 = 18, r = 0.5\)
   \(t_6 = 18(0.5)^5 = 4.5\)

\(r = \frac{-46}{23} = -2\)
   Use: \(t_n = t_1 r^{n-1}\)
   Substitute: \(n = 10, t_1 = 23, r = -2\)
   \(t_{10} = 23(-2)^{10-1} = 23(256) = 5884\)

\(r = \frac{1}{2} \div \frac{1}{8} = \frac{1}{4}\)
   Use: \(t_n = t_1 r^{n-1}\)
   Substitute: \(n = 5, t_1 = 2, r = \frac{1}{4}\)
   \(t_5 = 2 \left(\frac{1}{4}\right)^4 = \frac{1}{128}\)

6. Write the first 4 terms of each geometric sequence, given the 1st term and the common ratio. Identify the sequence as decreasing, increasing, or neither. Justify your answers.

a) \(t_1 = -3; \) \(r = 4\)
   \(t_1 = -3\)
   \(t_2 = (-3)(4) = -12\)
   \(t_3 = (-12)(4) = -48\)
   \(t_4 = (-48)(4) = -192\)
   The sequence is decreasing because the terms are decreasing.

b) \(t_1 = 5; \) \(r = 2\)
   \(t_1 = 5\)
   \(t_2 = (5)(2) = 10\)
   \(t_3 = (10)(2) = 20\)
   \(t_4 = (20)(2) = 40\)
   The sequence is increasing because the terms are increasing.
c) \( t_1 = -0.5; r = -3 \)
\[
\begin{align*}
t_1 &= -0.5 \\
t_2 &= (0.5)(-3) = 1.5 \\
t_3 &= (1.5)(-3) = -4.5 \\
t_4 &= (-4.5)(-3) = 13.5 \\
\end{align*}
\]
The sequence neither increases nor decreases because the terms alternate signs.

d) \( t_1 = \frac{1}{2}; r = \frac{2}{3} \)
\[
\begin{align*}
t_1 &= \frac{1}{2} \\
t_2 &= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{1}{3} \\
t_3 &= \left(\frac{3}{6}\right)\left(\frac{2}{3}\right) = \frac{2}{9} \\
t_4 &= \left(\frac{2}{9}\right)\left(\frac{2}{3}\right) = \frac{4}{27} \\
\end{align*}
\]
The sequence is decreasing because the terms are decreasing.

7. Write the first 5 terms of a geometric sequence where:

a) the 6th term is 64

Sample response:
\[
\begin{align*}
t_6 &= 64 \\
\text{Divide by a common ratio that is a factor of 64, such as } -2. \\
t_6 &= 64 \\
t_5 &= \frac{64}{-2} = -32 \\
t_4 &= \frac{-32}{-2} = 16 \\
t_3 &= 16 \\
t_2 &= \frac{16}{-2} = -8 \\
t_1 &= \frac{-8}{-2} = 4 \\
t_0 &= 4 \\
\end{align*}
\]
A possible geometric sequence is: 
\(-2, 4, -8, 16, -32, \ldots\)

b) the 1st term is \(\frac{3}{4}\)

Sample response:
\[
\begin{align*}
t_1 &= \frac{3}{4} \\
t_2 &= (\frac{3}{4})(4) = 3 \\
t_3 &= (3)(4) = 12 \\
t_4 &= (12)(4) = 48 \\
t_5 &= (48)(4) = 192 \\
\end{align*}
\]
A possible geometric sequence is: 
\(\frac{3}{4}, 3, 12, 48, 192, \ldots\)

c) every term is negative

Sample response:
\[
\begin{align*}
t_1 &= -5 \\
t_2 &= (5)(2) = -10 \\
t_3 &= (10)(2) = -20 \\
t_4 &= (20)(2) = -40 \\
t_5 &= (40)(2) = -80 \\
\end{align*}
\]
A possible geometric sequence is: 
\(-5, -10, -20, -40, -80, \ldots\)

d) every term is an even number

Sample response:
\[
\begin{align*}
t_1 &= 4 \\
t_2 &= (4)(3) = 12 \\
t_3 &= (12)(3) = 36 \\
t_4 &= (36)(3) = 108 \\
t_5 &= (108)(3) = 324 \\
\end{align*}
\]
A possible geometric sequence is: 
\(4, 12, 36, 108, 324, \ldots\)
8. Use the given data about each finite geometric sequence to determine the indicated values.

a) Given \( t_1 = -1 \) and \( r = -2 \)

i) Determine \( t_9 \).

Use: \( t_n = t_1r^{n-1} \)

Substitute: \( n = 9, t_1 = -1, r = -2 \)

\[ t_9 = (-1)(-2)^{9-1} \]

\[ t_9 = (-1)(-2)^8 \]

\[ t_9 = -256 \]

ii) The last term is \(-4096\). How many terms are in the sequence?

Use \( t_n = t_1r^{n-1} \) to determine \( n \).

Substitute: \( t_n = -4096, t_1 = -1, r = -2 \)

\[ -4096 = (-1)(-2)^{n-1} \quad \text{Divide by } -1. \]

\[ 4096 = (-2)^{n-1} \]

\[ (-2)^{12} = (-2)^{n-1} \quad \text{Equate exponents.} \]

\[ 12 = n - 1 \]

\[ n = 13 \]

There are 13 terms in the sequence.

b) Given \( t_1 = 0.002 \) and \( t_4 = 2 \)

i) Determine \( t_7 \).

Use \( t_n = t_1r^{n-1} \) to determine the common ratio, \( r \).

Substitute: \( n = 4, t_4 = 2, t_1 = 0.002 \)

\[ 2 = 0.002r^{4-1} \]

\[ 2 = 0.002r^3 \quad \text{Divide each side by } 0.002. \]

\[ \frac{1000}{2} = r^3 \]

\[ \sqrt[3]{1000} = r \]

\[ r = 10 \]

To determine \( t_7 \), use: \( t_n = t_1r^{n-1} \)

Substitute: \( n = 7, t_1 = 0.002, \text{ and } r = 10 \)

\[ t_7 = 0.002(10)^{7-1} \]

\[ t_7 = 0.002(10)^6 \]

\[ t_7 = 2000 \]

ii) Determine which term has the value 20 000.

Use \( t_n = t_1r^{n-1} \) to determine \( n \).

Substitute: \( t_n = 20000, t_1 = 0.002, r = 10 \)

\[ 20000 = 0.002(10)^{n-1} \]

\[ 20000 = 2(10)^{n-1} \]

\[ 10000 = 10^{n-1} \]

\[ 10^3 = 10^{n-1} \]

\[ 7 = n - 1 \]

\[ n = 8 \]

20 000 is the 8th term.
9. A ball is dropped from a height of 25 m. After each bounce, the ball rises to 80% of the previous height.

a) Write the first 3 terms of a geometric sequence that models the height of the ball in metres.

\[ 80\% = 0.8, \text{ so } r = 0.8 \]
\[ t_1 = 25(0.8), \text{ or } 20 \]
\[ t_2 = 20(0.8), \text{ or } 16 \]
\[ t_3 = 16(0.8), \text{ or } 12.8 \]

The geometric series is 20, 16, 12.8, \ldots

b) To the nearest centimetre, to what height does the ball rise after the 5th bounce?

\[ \text{After the 4th bounce, the height is: } 12.8(0.8) = 10.24 \]
\[ \text{After the 5th bounce, the height is: } 10.24(0.8) = 8.192 \]

After the 5th bounce, the ball rises to a height of 8.19 m.

c) To the nearest centimetre, to what height does the ball rise after the 10th bounce?

Use \( t_n = t_1r^{n-1} \) to determine \( t_{10} \).
Substitute: \( r = 0.8, t_1 = 20, n = 10 \)
\[ t_{10} = 20(0.8)^{10-1} \]
\[ t_{10} = 20(0.8)^9 \]
\[ t_{10} = 2.6843 \ldots \]

After the 10th bounce, the ball rises to a height of 2.68 m.

d) After how many bounces does the ball rise to a height less than 1 m?

Use \( t_n = t_1r^{n-1} \) to determine \( n \).
Substitute: \( r = 0.8, t_1 = 20, t_n = 1 \)
\[ 1 = 20(0.8)^{n-1} \quad \text{Divide each side by } 20. \]
\[ 0.05 = (0.8)^{n-1} \quad \text{Use guess and test.} \]
\[ (0.8)^{13} = 0.0549 \ldots \]
\[ (0.8)^{14} = 0.0439 \ldots \]

When \( n = 14; \) that is, the 15th bounce, the ball rises to a height less than 1 m.

10. Suppose a person is given 1¢ on the first day of April; 3¢ on the second day; 9¢ on the third day, and so on. This pattern continues throughout April.

a) About how much money will the person be given on the last day of April?

There are 30 days in April.
The daily amounts, in cents, form this geometric sequence:
\[ 1, 1(3), 1(3)^2, \ldots, 1(3)^{29} \]
The amount on the last day, in cents, is \[ 1(3)^{29} = 6.863 \times 10^{13} \]
Divide by 100 to convert the amount to dollars: approximately $6.863 \times 10^{11}$

b) Why might it be difficult to determine the exact amount using a calculator?

A calculator screen shows only 10 digits, and the number of digits in the amount of money in dollars is greater than 10.
11. In a geometric sequence, \( t_3 = 9 \) and \( t_6 = 1.125 \); determine \( t_7 \) and \( t_9 \).

Use \( t_n = t_1 r^{n-1} \) twice to get two equations.

For \( t_3 \), substitute: \( n = 3, t_3 = 9 \)

\[ 9 = t_1 r^2 \] ①

For \( t_6 \), substitute: \( n = 6, t_6 = 1.125 \)

\[ 1.125 = t_1 r^5 \] ②

Write equation ② as:

\[ 1.125 = t_1 r^5 = t_1 r^2(r^3) \]

From equation ①, substitute \( t_1 r^2 = 9 \)

Divide by 9.

\[ 0.125 = r^3 \] ③

\[ r = \sqrt[3]{0.125} \]

\[ r = 0.5 \]

So, \( t_7 = t_1 r \) and, \( t_9 = t_1 r^2 \)

\[ t_7 = 1.125(0.5) \]

\[ t_9 = 0.5625(0.5)^2 \]

\[ t_7 = 0.5625 \]

\[ t_9 = 0.140625 \]

12. A beekeeper starts her business with 200 bees. New bees are hatched at a rate of 104% each week. How many bees were there after week 15?

Use \( t_n = t_1 r^{n-1} \) to determine \( t_{15} \).

Substitute: \( r = 1.04, t_1 = 200, n = 15 \)

\[ t_{15} = 200(1.04)^{15-1} \]

\[ t_{15} = 200(1.04)^{14} \]

\[ t_{15} = 346.3352 \ldots \]

After week 15, there were about 346 bees.

13. A ream of paper is about 2 in. thick. Imagine a ream of paper that is repeatedly cut in half and the two halves stacked one on top of the other. How many cuts have to be made before the stack of paper is taller than 318 ft., the height of Le Chateau York in Winnipeg, Manitoba? Justify your answer.

Let the number of cuts be \( n \).

The heights of the stacks of paper form, in inches, a geometric sequence with 1st term 2 and common ratio 2:

\[ 2, \ 2(2), \ 2(2)^2, \ 2(2)^3, \ldots \ 2(2)^n \]

Write 318 ft. in inches: 318(12 in.) = 3816 in.

Write an equation:

\[ 2(2)^n = 3816 \]

Solve for \( n \).

\[ 2^n = 1908 \]

Use guess and test: \( 2^{10} = 1024; 2^{11} = 2048 \)

10 cuts will not be enough.

11 cuts will produce a stack that is: \( 2(2)^{11} \) in. = 4096 in. high

11 cuts have to be made.
14. Between the Canadian censuses in 2001 and 2006, the number of people who could converse in Cree had increased by 7%. In 2006, 87 285 people could converse in Cree. Assume the 5-year increase continues to be 7%. Estimate to the nearest hundred how many people will be able to converse in Cree in 2031.

To model a growth rate of 7%, multiply by 1.07.
The number of people every 5 years form a geometric sequence with first term 87 285 and common ratio 1.07.
Every 5 years is: 2006, 2011, 2016, 2021, 2026, 2031, . . .
So, the number of people in 2031 is:

\[ 87 \, 285 \times 1.07^5 = 122 \, 421.7278 . . . \]

The number of people who will be able to converse in Cree in 2031 will be approximately 122 400.

15. A farmer in Saskatchewan wants to estimate the value of a new combine after several years of use. A new combine worth $370 000 depreciates in value in about 10% each year.

a) Estimate the value of the combine at the end of each of the first 5 years. Write the values as a geometric sequence.

When the value decreases by 10%, the new value is 90% of the original value.
To determine a depreciation value of 10%, multiply by 0.9.
The values, in dollars, at the end of each of the first 5 years are:

\[ 370 \, 000 \times 0.9, \ 370 \, 000 \times 0.9^2, \ 370 \, 000 \times 0.9^3, \ 370 \, 000 \times 0.9^4, \ 370 \, 000 \times 0.9^5 \]
The values, to the nearest dollar, are:

$333 000, $299 700, $269 730, $242 757, $218 481

b) Predict the value of the combine at the end of 10 years.

At the end of 10 years, to the nearest dollar, the value is:

\[ 370 \, 000 \times 0.9^{10} = $129 \, 011 \]

16. a) Show that squaring each term in a geometric sequence produces the same type of sequence. What is the common ratio?

Consider the sequence:

\[ t, \ tr, \ tr^2, \ tr^3, \ r^4, \ldots, \ t, r^{n-1} \]

Square each term. The new sequence is:

\[ t^2, \ tr^2, \ tr^4, \ tr^6, \ r^8, \ldots, \ t^2 r^{2n-2} \]

This is a geometric sequence with 1st term \( t_1^2 \) and common ratio \( r^2 \).
b) Show that raising each term in a geometric sequence to the $m$th power of each term produces the same type of sequence. What is the common ratio?

Consider the sequence: $t, t\cdot r, t\cdot r^2, t\cdot r^3, \ldots, t\cdot r^{m-1}$
Raise each term to the $m$th power.
The new sequence is: $t^m, t^m\cdot r^m, t^m\cdot r^{2m}, t^m\cdot r^{3m}, \ldots, t^m\cdot r^{mn-m}$
This is a geometric sequence with 1st term $t^m$ and common ratio $r^m$. 